CHAPTER-5

NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Structure

5.1 Definition: Let U is an open sunset of R^n, $x=(x_1,...,x_n)\in R^n$ and let $u:\bar{U}\subseteq R^n\to R$. A general form of first-order partial differential equation for $u=u(x)$ is given by

$$F(Du,u,x)=0, \qquad \dots (1)$$

where $F:R^n\times R\times\bar{U}\to R$ is a given function, Du is the vector of partial derivatives of u and $u(x)$ is the unknown function.

We can write equation (1) as

$$\begin{aligned} F &= F(p,z,x) \\ &= F(p_{1,}p_2...,p_{n,},z,x_1,x_2,...,x_n) \end{aligned}$$

for $p\in R^n$, $z\in R$, $x\in U$.

Here, "p" is the name of the variable for which we substitute the gradient Du and "z" is the variable for which we substitute $u(x)$. We also assume hereafter that F is smooth, and set

$$\begin{aligned} D_pF &= \left(F_{p_1},F_{p_2},...,F_{p_n}\right) \\ D_zF &= F_z \\ D_x &= \left(F_{x_1},F_{x_2},...,F_{x_n}\right) \end{aligned}.$$

Remark: The PDE $F(Du,u,x)=0$ is usually accompanied by a boundary condition of the form $u=g$ on ∂U. Such a problem is usually called a boundary value problem. Here our main concern is to search solution for the non-linear PDE

Complete Integral: Consider the non-linear first order PDE

$$F(Du,u,x)=0 \qquad \text{… (1)}$$

Suppose first that $A\subset R^n$ is an open set. Assume for each parameter $a=(a_1,...,a_n)\in A$, we have a C^2 solution

$$u=u(x;a) \qquad \text{… (2)}$$

of the PDE (1) and

$$\left(D_a u, D_{xa}^2 u\right)=\begin{bmatrix} u_{a_1} & u_{x_1a_1} & \cdots & u_{x_na_1} \\ u_{a_2} & u_{x_1a_2} & \cdots & u_{x_na_2} \\ \cdots & \cdots & \cdots & \cdots \\ u_{a_n} & u_{x_1a_n} & \cdots & u_{x_na_n} \end{bmatrix} \qquad \text{… (3)}$$

A C^2 function $u=u(x;a)$ (shown in equation (2)) is called a complete integral in $U\times A$ provided

(i) $u(x;a)$ solves the PDE(1) for each $a\in A$

(ii) $rank\left(D_a u, D_{xa}^2 u\right)=n \quad (x\in U, a\in A)$

Note: Condition (ii) ensures $u(x;a)$"depends on all the n independent parameters $a_1,...,a_n$".

Example 1: The eikonal equation,

$$|Du|=1 \qquad \text{… (4)}$$

Introduced by Hamilton in 1827 is an approximation to the equations which govern the behaviour of light travelling through varying materials. A solution, depending on parameters $\|a\|=1, b\in R$ is

$$u(x;a,b)=a.x+b \qquad \text{… (5)}$$

Example 2: The Clairaut's equation is the PDE

$$x.Du+f(Du)=u \qquad \text{… (6)}$$

where $f:R^n\to R$ is given.

A complete integral is

$$u(x;a)=a\bullet x+f(a) \qquad (x\in U) \qquad \text{… (7)}$$

for $a\in R^n$.

Example 3: The Hamilton-Jacobi Equation

$$u_t + H(Du) = 0 \qquad \ldots (8)$$

with $H : R^n \to R$ is given and $u = u(x,t) : R^n \times R \to R$ .A solution depending on parameters $a \in R^n, b \in R$ is

$$u(x,t;a,b) = a \bullet x - tH(a) + b \qquad \ldots (9)$$

where $t \geq 0$.

Remark: For simplicity, in most of what follows, we restrict to $n = 2$. We call the two variables x, y. Thus, we reduce to the case

$$F(u_x, u_y, u, x, y) = 0 \qquad \ldots (7)$$

In this case, the solution $u = u(x, y)$ is a surface in R^3. The normal direction to the surface at each point is given by the vector $(u_x, u_y, -1)$.

5.2 Envelope

Definition: Let $u = u(x;a)$ be a C^1 function of x and U and A are open subsets of R^n. Consider the vector equation

$$D_a u(x;a) = 0 \quad (x \in U, a \in A) \qquad \ldots (1)$$

Suppose that we can solve (1) for the parameter a as a C^1 function of x,

$$a = \phi(x) \qquad \ldots (2)$$

Thus

$$D_a u(x;\phi(x)) = 0 \quad (x \in U) \qquad \ldots (3)$$

We can call

$$v(x) := u(x;\phi(x)) \quad (x \in U) \qquad \ldots (4)$$

is the envelope of the function $\{u(.;a)\}_{a \in A}$

Remarks: We can build new solution of nonlinear first order PDE by forming envelope and such types of solutions are called singular integral of the given PDE.

Theorem: Construction of new solutions

Suppose for each $a \in A$ as above that $u = u(.;a)$ solves the partial differential equation

$$F(Du, u, x) = 0 \qquad \ldots (5)$$

Assume further that the envelope v, defined (3) and (4) above, exists and is a C^1 function. Then v solves (5) as well.

Proof: We have $v(x)=u(x;\phi(x))$

$$v_{x_i}(x)=u_{x_i}(x;\phi(x))+\sum_{j=1}^{m}u_{a_j}(x,\phi(x))\phi_{x_i}^{j}(x)$$

$$=u_{x_i}(x;\phi(x))$$

for $i=1,\ldots,n$.

Hence for each $x\in U$,

$$F(Dv(x),v(x),x)=F(Du(x;\phi(x)),u(x;\phi(x)),x)=0$$

Note: The geometric idea is that for each $x\in U$, the graph of v is tangent to the graph of $u(.;a)$ for $a=\phi(x)$. Thus $Dv=D_x u(.;a)$ at x, for $a=\phi(x)$.

Example 4: Consider the PDE

$$u^2\left(1+|Du|^2\right)=1 \qquad \ldots (6)$$

The complete integral is

$$u(x,a)=\pm\left(1-|x-a|^2\right)^{1/2} \qquad (|x-a|<1)$$

We find that

$$D_a u=\frac{\mp(x-a)}{\left(1-|x-a|^2\right)^{1/2}}=0$$

provided $a=\phi(x)=x$.

Thus $v\equiv\pm1$ are singular integrals of (6).

5.3 Characteristics

Theorem: Structure of Characteristics PDE

Let $u\in C^2(U)$ solves the non-linear PDE

$$F(Du,u,x)=0 \text{ in } U$$

Assume $\bar{x}(.)=(x^1,x^2,\ldots,x^n)$ solves the ODE $\dot{\bar{x}}=D_pF\left(\overline{p(s)},z(s),\overline{x(s)}\right)$,

where

$\overline{p(s)} = Du\left(\bar{x}(.)\right), \qquad z(s) = u(\bar{x}(.))$

Then $\bar{p}(.)$ solves the ODE.

$\dot{\bar{p}} = -D_x F\left(\overline{p(s)}, z(s), \overline{x(s)}\right) - D_z F\left(\overline{p(s)}, z(s), \overline{x(s)}\right)\overline{p(s)}$ (3)

and *z(s)* solves the ODE $\dot{z}(s) = D_p F\left(\overline{p(s)}, z(s), \overline{x(s)}\right).\overline{p(s)}$ for those *s* such that $\bar{x}(s) \in U$

Proof: Consider nonlinear first order PDE

$$F\left(Du, u, x\right) = 0 \text{ in } U \qquad \dots (1)$$

subject now to the boundary condition

$$u = g \quad \text{on } \Gamma \qquad \dots (2)$$

where $\Gamma \subseteq \partial U$ and $g : \Gamma \to R$ are given.

We suppose that F and g are smooth functions. Now we derive the method of characteristics which solves (1) and (2) by converting PDE into appropriates system of ODE. Initially, we would like to calculate u(x) by finding some curve lying within U, connecting x with a point $x_0 \in \Gamma$ and along which we can calculate u. Since equation (2) says $u = g$ on Γ. So we know the value of u at one end x_0 and we hope then to able to find the value of u all along the curve, and also at the particular point x.

Let us suppose the curve is described parametrically by the function

$\bar{x}(s) = \left(x^1(s), \dots, x^n(s)\right)$, the parameter s lying in some subinterval of R

Assuming u is a C^2 solution of (1), we define

$$z(s) = u\left(\bar{x}(s)\right) \qquad \dots (3)$$

Set

$$\bar{p}(s) = Du\left(\bar{x}(s)\right) \qquad \dots (4)$$

i.e. $\bar{p}(s) = \left(p^1(s), \dots, p^n(s)\right)$, where

$$p^i(s) = u_{x_i}\left(\bar{x}(s)\right) \quad \left(i = 1, \dots, n\right). \qquad \dots (5)$$

So $z(.)$ gives the values of u along the curve and $\bar{p}(.)$ records the values of the gradient Du.

First we differentiate (5)

$$\dot{p}^i(s) = \sum_{j=1}^{n} u_{x_i x_j}\left(\bar{x}(s)\right)\dot{x}^j(s) \qquad \dots (6)$$

where $\bullet = \frac{d}{ds}$

We can also differentiate the PDE (1) with respect to x

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}(Du,u,x) u_{x_j x_i}\left(\bar{x}(s)\right)\dot{x}^j(s) + \frac{\partial F}{\partial z}(Du,u,x)u_{x_i} + \frac{\partial F}{\partial x_i}(Du,u,x) = 0 \quad \ldots (7)$$

We set

$$\dot{x}^j(s) = \frac{\partial F}{\partial p_j}(\bar{p}(s), z(s), \bar{x}(s)) \qquad (j = 1, 2, \ldots, n) \qquad \ldots (8)$$

Assuming (8) holds, we evaluate (7) at $x = \bar{x}(s)$ and using equations (3) and (4), we have the identity

$$\sum_{j=1}^{n} \frac{\partial F}{\partial p_j}\left(\bar{p}(s), z(s), \bar{x}(s)\right) u_{x_i x_j}\left(\bar{x}(s)\right) + \frac{\partial F}{\partial z}\left(\bar{p}(s), z(s), \bar{x}(s)\right) p^i(s) + \frac{\partial F}{\partial x_i}\left(\bar{p}(s), z(s), \bar{x}(s)\right) = 0$$

Put this expression and (8) into (6)

$$\dot{p}^i(s) = -\frac{\partial F}{\partial x_i}\left(\bar{p}(s), z(s), \bar{x}(s)\right) - \frac{\partial F}{\partial z}\left(\bar{p}(s), z(s), \bar{x}(s)\right) p^i(s) \qquad \ldots (9)$$

Lastly, we differentiate (3)

$$\dot{z}(s) = \sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\left(\bar{x}(s)\right)\dot{x}^j(s) = \sum_{j=1}^{n} p^j(s) \frac{\partial F}{\partial p_j}\left(\bar{p}(s), z(s), \bar{x}(s)\right) \qquad \ldots (10)$$

the second equality holding by (5) and (8). We summarize by rewriting equation (8)-(10) in vector notation

$$\begin{aligned}
\dot{\bar{p}}(s) &= -D_x F\left(\bar{p}(s), z(s), \bar{x}(s)\right) - D_x F\left(\bar{p}(s), z(s), \bar{x}(s)\right).\bar{p}(s) \\
\dot{z}(s) &= D_p F\left(\bar{p}(s), z(s), \bar{x}(s)\right).\bar{p}(s) \qquad \ldots (11) \\
\dot{\bar{x}}(s) &= D_p F\left(\bar{p}(s), z(s), \bar{x}(s)\right)
\end{aligned}$$

This system of 2n+1 first order ODE comprises the **characteristic equation of the nonlinear first order PDE (1).**

The functions $\bar{p}(.) = \left(p^1(.), \ldots, p^n(.)\right), z(.), \bar{x}(.) = \left(x^1(.), \ldots, x^n(.)\right)$ are called the **characteristics.**

Remark: The characteristics ODE are truly remarkable in that they form a closed system of equations for $\bar{x}(.), z(.) = u\left(\bar{x}(.)\right)$ and $\bar{p}(.) = Du\left(\bar{x}(.)\right)$, whenever u is a smooth solution of the general nonlinear PDE(1).

We can use $X(s)$ in place of $\bar{x}(s)$.

Now we discuss some special cases for which the structure of characteristics equations is especially simple.

(a) Article

Let us consider the PDE of the form $F(Du,u,x)=0$ to be linear and homogeneous and thus has the form

$$F(Du,u,x)=b(x).Du(x)+c(x)u(x)=0 \qquad (x\in U) \qquad \dots (1)$$

Equation (1) can be written as

$$F(p,z,x)=b(x).p+c(x)z$$

So characteristics equations are

$$\dot{\bar{x}}(s)=D_pF=b(x)$$

$$=b(\bar{x}(s)) \qquad \text{(From last expression)}$$

and $$\dot{z}(s)=D_pF.\bar{p}=b(\bar{x}(s)).\bar{p}(s) \qquad \text{(From last expression)}$$

$$=-c(\bar{x}(s))z(s)$$

Thus

$$\begin{cases} \dot{\bar{x}}(s)=b(\bar{x}(s)) \\ \dot{z}(s)=-c(\bar{x}(s))z(s) \end{cases} \qquad \dots (2)$$

comprise the characteristics equations for the linear first order PDE(1).

Example 5: Solve two dimensional system

$$\begin{cases} x_1u_{x_2}-x_2u_{x_1}=u \ \ in\, U \\ \qquad u=g \qquad on\,\Gamma \end{cases} \qquad \dots (3)$$

where U is the quadrant $\{x_1>0, x_2>0\}$ and $\Gamma=\{x_1>0, x_2=0\}\subseteq \partial U$.

Solution: Comparing (3) with (1), we have

$$F(Du,u,x)=x_1u_{x_2}-x_2u_{x_1}-u=0$$

$$\Rightarrow (-x_2,x_1).(u_{x_1},u_{x_2})-u=0$$

We get,

$$b(\bar{x}(s))=(-x_2,x_1), \qquad c(\bar{x}(s))=-1$$

Now $$b(\bar{x}(s))=(b_1(x),b_2(x))$$

$$= \left(-x_2, x_1\right)$$

$$\Rightarrow b_1\left(x\right) = -x_2, b_2\left(x\right) = x_1$$

The characteristics equations are

$$\dot{X}\left(s\right) = b\left(X\left(s\right)\right)$$

and $$\dot{z}\left(s\right) = -c\left(X\left(s\right)\right)z\left(s\right)$$

Therefore $$\dot{z}\left(s\right) = z\left(s\right)$$

$$\dot{X}\left(s\right) = \left(-x_2\left(s\right), x_1\left(s\right)\right)$$

$$\Rightarrow \left(\dot{x}_1\left(s\right), \dot{x}_2\left(s\right)\right) = \left(-x_2\left(s\right), x_1\left(s\right)\right)$$

$$\Rightarrow \dot{x}_1\left(s\right) = -x_2\left(s\right) \text{ and } \dot{x}_2\left(s\right) = x_1\left(s\right) \qquad \ldots(4)$$

Now $$\ddot{x}_1\left(s\right) = -\dot{x}_2\left(s\right) = -x_1\left(s\right)$$

$$\Rightarrow \ddot{x}_1\left(s\right) + x_1\left(s\right) = 0$$

Auxiliary equation is $D^2 + 1 = 0$

$$\Rightarrow D = \pm i$$

$$\Rightarrow x_1\left(s\right) = c_1 \cos s + c_2 \sin s \qquad \ldots (5)$$

So $$\dot{x}_2\left(s\right) = c_1 \cos s + c_2 \sin s \qquad \ldots(6)$$

Integrate (5) w.r.t.s

$$x_2\left(s\right) = c_1 \sin s - c_2 \cos s + c_3 \qquad \ldots(7)$$

From (5), we have

$$\dot{x}_1\left(s\right) = -c_1 \sin s + c_2 \cos s \qquad \ldots(8)$$

Comparing (4) and (8)

$$-x_2\left(s\right) = -c_1 \sin s + c_2 \cos s$$

$$\Rightarrow x_2\left(s\right) = c_1 \sin s - c_2 \cos s \qquad \ldots(9)$$

From (7) and (9)

$$c_3 = 0$$

Therefore $$x_2(s) = c_1 \sin s - c_2 \cos s \qquad \ldots(10)$$

Taking $s = 0$ in (10)

$$x_2(0) = -c_2$$

$$\Rightarrow c_2 = 0 \qquad \left[\Gamma = \left\{(x_1(s), x_2(s)) \middle| x_2 = 0 \, at \quad s = 0\right\}\right]$$

Therefore $$x_1(s) = c_1 \cos s \qquad \ldots(11)$$

and $$x_2(s) = c_1 \sin s \qquad \ldots(12)$$

Put $s = 0$ in (11)

$$x_1(0) = c_1$$

Let $x^0 = x_1(0) = c_1$

Put value of $c_1 = x^0$ in (11) and (12)

$$x_1(s) = x^0 \cos s$$
$$x_2(s) = x^0 \sin s$$

Also we have

$$\dot{z}(s) = z(s)$$
$$\Rightarrow \frac{dz}{ds} = z(s)$$

Integrating w.r.t.s

$$\log z = s + \log z^0$$
$$\Rightarrow \log \frac{z}{z^0} = s$$
$$\Rightarrow z = z^0 e^s$$
$$\Rightarrow z(0) = z^0$$

Therefore $$z(s) = z(0)e^s$$

Also $$u = g \; on \quad \Gamma$$

$$\Rightarrow u(x(s), 0) = g(x_1(s)) \qquad \ldots(13)$$

We know that $u(x(s)) = z(s)$

So
$$u\left(x_1(s), x_2(s)\right) = z(s)$$
$$\Rightarrow u\left(x_1(0), 0\right) = z(0) = z^0 \qquad \ldots(14)$$

Put (14) in (13)
$$z(s) = g\left(x^0\right)e^s$$

Thus we have
$$x_1(s) = c_1 \cos s = x^0 \cos s$$
and
$$x_2(s) = c_1 \sin s = x^0 \sin s$$
and
$$z(s) = g\left(x^0\right)e^s$$

Now select s>0 and $x^0 > 0$, so that
$$\left(x_1, x_2\right) = \left(x_1(s), x_2(s)\right) = \left(x^0 \cos s, x^0 \sin s\right)$$
$$\Rightarrow x_1 = x^0 \cos s \; and \quad x_2 = x^0 \sin s$$

Consider,
$$x_1^2 + x_2^2 = x^{0^2}\left(\sin^2 s + \cos^2 s\right) = x^{0^2}$$
$$\Rightarrow \sqrt{x_1^2 + x_2^2} = x^0$$

We have
$$\tan s = \frac{x_2}{x_1}$$
$$\Rightarrow s = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

Thus
$$u\left(x(s)\right) = z(s) = g\left(x^0\right)e^s$$
$$\Rightarrow u\left(x(s)\right) = g\left(\sqrt{x_1^2 + x_2^2}\right)e^{\arctan\left(\frac{x_2}{x_1}\right)}$$

which is the required solution.

(b) Article

A quasilinear PDE is of the form
$$F\left(Du, u, x\right) = b\left(x, u(x)\right).Du(x) + c\left(x, u(x)\right) = 0 \qquad \ldots \; (1)$$

Equation (1) can be written as

$$F(p,z,x)=b(x,z).p+c(x,z)$$

Now $$D_pF=b(x,z)$$

Thus the characteristic equations becomes

$$\dot{X}(s)=D_pF=b(X(s),z(s))$$

and $$\dot{z}(s)=D_pF.\overline{p}$$

$$=b(X(s),z(s)).\overline{p}(s)$$
$$=-c(X(s),z(s))$$

Consequently

$$\begin{cases}\dot{X}(s)=b(X(s),z(s))\\ \dot{z}(s)=-c(X(s),z(s))\end{cases} \quad ..(2)$$

are the characteristic equations for the quasilinear first order PDE (1).

Example 6: Consider a boundary-value problem for a semilinear PDE

$$\begin{cases}u_{x_1}+u_{x_2}=u^2 \ in\, U\\ \quad u=g \quad on\,\Gamma\end{cases} \quad \ldots(3)$$

where U is half-space $\{x_2>0\}$ *and* $\Gamma=\{x_2=0\}=\partial U$.

Solution: Comparing (3) with (1), we have

$$b=(1,1) \text{ and } c=-z^2$$

Then (2) becomes

$$\begin{cases}\dot{x}^1=1,\dot{x}^2=1\\ \quad \dot{z}=z^2\end{cases}$$

Consequently

$$\begin{cases}x^1(s)=x^0+s,x^2(s)=s\\ z(s)=\dfrac{z^0}{1-sz^0}=\dfrac{g(x^0)}{1-sg(x^0)}\end{cases}$$

where $x^0 \in R, s \geq 0$, provided the denominator is not zero.

Fix a point $(x_1, x_2) \in U$. We select $s > 0$ and $x^0 \in \boldsymbol{R}$, so that $(x_1, x_2) = (x^1(s), x^2(s)) = (x^0 + s, s)$

i.e. $x^0 = x_1 - x_2, s = x_2$.

Then

$$u(x_1, x_2) = u(x^1(s), x^2(s)) = z(s) = \frac{g(x^0)}{1 - sg(x^0)}$$

$$= \frac{g(x_1 - x_2)}{1 - x_2 g(x_1 - x_2)}, 1 - x_2 g(x_1 - x_2) \neq 0$$

which is the required solution.

(c) In this case, we will discuss about characteristics equation of fully nonlinear PDE.

Example 7: Consider the fully nonlinear problem

$$\begin{cases} u_{x_1} u_{x_2} = u \ in\, U \\ u = x_2^2 \quad on\, \Gamma \end{cases} \qquad ..(1)$$

where $U = \{x_1 > 0\}, \Gamma = \{x_1 = 0\} = \partial U$

Here $F(p, z, x) = p_1 p_2 - z$. Then the characteristic equations becomes

$$\begin{cases} \dot{p}^1 = p^1, \dot{p}^2 = p^2 \\ \dot{z} = 2p^1 p^2 \\ \dot{x}^1 = p^2, \dot{x}^2 = p^1 \end{cases}$$

We integrate these equations and we find

$$\begin{cases} x^1(s) = p_2^0(e^s - 1), x^2(s) = x^0 + p_1^0(e^s - 1) \\ z(s) = z^0 + p_1^0 p_2^0 (e^{2s} - 1) \\ p^1(s) = p_1^0 e^s, p^2(s) = p_2^0 e^s \end{cases}$$

Since $u = x_2^2$ on Γ, $p_2^0 = u_{x_2}(0, x^0) = 2x^0$.

Therefore, the PDE $u_{x_1} u_{x_2} = u$ itself implies $p_1^0 p_2^0 = z^0 = (x^0)^2$, and so $p_1^0 = \dfrac{x^0}{2}$.

Thus we have,

$$\begin{cases} x^1(s) = 2x^0\left(e^s - 1\right), x^2(s) = \dfrac{x^0}{2}\left(e^s + 1\right) \\ z(s) = \left(x^0\right)^2 e^{2s} \\ p^1(s) = \dfrac{x^0}{2}e^s, \, p^2(s) = 2x^0 e^s \end{cases}$$

Fix a point $(x_1, x_2) \in U$. Choose s and x^0 so that $(x_1, x_2) = (x^1(s), x^2(s)) = (2x^0(e^s - 1), \frac{x^0}{2}(e^s + 1))$

and so

$$u(x_1, x_2) = u\left(x^1(s), x^2(s)\right) = z(s) = \left(x^0\right)^2 e^{2s}$$

$$= \frac{\left(x_1 + 4x_2\right)^2}{16}$$

Exercise:

1. Find the characteristics of the following equations:
   (a) $x_1 u_{x_1} + x_2 u_{x_2} = 2u, \;\; u(x_1, 1) = g(x_1)$
   (b) $u_t + b.Du = f \quad in \quad R^n \times (0, \infty), b \in R^n, \;\; f = f(x,t)$
2. Prove that the characteristics for the Hamiltonian-Jacobi equation
   $u_t + H(Du, x) = 0$
   are

$$\dot{\overline{p}}(s) = -D_x H\left(\overline{p}(s), \overline{x}(s)\right)$$
$$\dot{z}(s) = D_p H\left(\overline{p}(s), \overline{x}(s)\right).\overline{p}(s) - H\left(\overline{p}(s), \overline{x}(s)\right)$$
$$\dot{\overline{x}}(s) = D_p H\left(\overline{p}(s), \overline{x}(s)\right)$$

5.4 Hamilton-Jacobi Equation

The initial-value problem for the Hamilton-Jacobi equation is

$$\begin{cases} u_t + H(Du) = 0 \; in \;\; R^n \times (0, \infty) \\ \qquad u = g \qquad on \, R^n \times \{t = 0\} \end{cases}$$

Here $u : R^n \times [0, \infty) \to R$ is the unknown, $u = u(x,t)$, and $Du = D_x u = \left(u_{x_1}, ..., u_{x_n}\right)$. The Hamiltonian $H : R^n \to R$ and the initial function $g : R^n \to R$ are given.

Note: Two characteristic equations associated with the Hamilton-Jacobi PDE

$$u_t + H(Du, x) = 0$$

are Hamilton's ODE

$$\begin{cases} \dot{\overline{x}} = D_p H\left(\overline{p}(s), \overline{x}(s)\right) \\ \dot{\overline{p}} = -D_x H\left(\overline{p}(s), \overline{x}(s)\right) \end{cases}$$

which arise in the classical calculus of variations and in mechanics.

5.4.1 Derivation of Hamilton's ODE from a Variational Principle (Calculus of Variation)

Article: Suppose that $L : R^n \times R^n \to R$ is a given smooth function, which is called Lagrangian.

We write

$$L = L(q, x) = L(q_1, ..., q_n, x_1, ..., x_n)$$

and

$$\begin{cases} D_q L = \left(L_{q_1} ... L_{q_n}\right) \\ D_x L = \left(L_{x_1} ... L_{x_n}\right) \end{cases}$$

Where $q, x \in R^n$

For any two fix points $x, y \in R^n$ and a time $t > 0$ and we introduce the action functional

$$I\left[\overline{w}(.)\right] = \int_0^t L\left(\dot{\overline{w}}(s), \overline{w}(s)\right) ds \qquad \dots (2)$$

where the functions $\overline{w}(.) = \left(w^1(.), w^2(.), ..., w^n(.)\right)$ belonging to the admissible class

$$A = \left\{\overline{w}(.) \in C^2\left([0,t]; R^n\right) \middle| \overline{w}(0) = y, \overline{w}(t) = x\right\}$$

Thus, a C^2 curve $\overline{w}(.)$ belongs to A if it starts at the point y at time 0 and reaches the point x at time t. According to the calculus of variations, we shall find a parametric curve $\overline{x}(.) \in A$ such that

$$I\left[\overline{x}(.)\right] = \min_{\overline{w}(.) \in A} I\left[\overline{w}(.)\right] \qquad \dots (3)$$

i.e., we are seeking a function $\overline{x}(.)$ which minimizes the functional $I[.]$ among all admissible candidates $\overline{w}(.) \in A$.

5.4.2 Theorem: Euler-Lagrange Equations

Prove that any minimizer $\bar{x}(.)\in A$ of $I[\bullet]$ solves the system of Euler-Lagrange equations

$$(4) \qquad -\frac{d}{ds}\left(D_q L\left(\dot{\bar{x}}(s),\bar{x}(s)\right)\right)+D_x L\left(\dot{\bar{x}}(s)\right) \qquad (0\le s\le t)$$

Proof: Consider a smooth function $\bar{v}:[0,t]\to R^n$ satisfying

$$\bar{v}(0)=\bar{v}(t)=0 \qquad \dots (5)$$

and $\bar{v}=\left(v^1,...,v^n\right)$

For $c\in R$, we define

$$\bar{w}(.)=\bar{x}(.)+c\bar{v}(.) \qquad \dots (6)$$

Then, $\bar{w}(.)$ belongs to the admissible class A and $\bar{x}(.)$ being the minimizer of the action functional and so

$$I\left[\bar{x}(.)\right]\le I\left[\bar{w}(.)\right]$$

Therefore the real-valued function

$$i(c)=I\left[\bar{x}(.)+c\bar{v}(.)\right]$$

Has a minimizer at $c=0$ and consequently

$$i'(0)=0 \qquad \dots (7)$$

provided $i'(0)$ exists.

Next we shall compute this derivative explicitly and we get

$$i(c)=\int_0^t L\left(\dot{\bar{x}}(s)+c\dot{\bar{v}}(s),\bar{x}(s)+c\bar{v}(s)\right)ds$$

And differentiating above equation w.r.t. c, we obtain

$$i'(c)=\int_0^t\sum_{i=1}^n L_{q_i}\left(\dot{\bar{x}}+c\dot{\bar{v}},x+c\bar{v}\right)\dot{v}^i+L_{x_i}\left(\dot{\bar{x}}+c\dot{\bar{v}},x+c\bar{v}\right)v^i ds$$

Set $c=0$ and using (7), we have

$$0=i'(0)=\int_0^t\sum_{i=1}^n L_{q_i}\left(\dot{\bar{x}},\bar{x}\right)\dot{v}^i+L_{x_i}\left(\dot{\bar{x}},\bar{x}\right)v^i ds \qquad \dots (8)$$

Now we integrate (8) by parts in the first term inside the integral and using (5), we have

$$0=\sum_{i=1}^{n}\int_{0}^{t}\left[-\frac{d}{ds}\left(L_{q_i}\left(\dot{\bar{x}},\bar{x}\right)\right)+L_{x_i}\left(\dot{\bar{x}},\bar{x}\right)\right]v^i ds$$

This identity is valid for all smooth functions $\bar{v}=\left(v^1,...,v^n\right)$ satisfying (5) and so

$$-\frac{d}{ds}\left(L_{q_i}\left(\dot{\bar{x}},\bar{x}\right)\right)+L_{x_i}\left(\dot{\bar{x}},\bar{x}\right)=0$$

for $0\le s\le t, i=1,...,n$

Remark: We see that any minimizer $\bar{x}(.)\in A$ of $I[.]$ solves the Euler-Lagrange system of ODE. It is also possible that a curve $\bar{x}(.)\in A$ may solve the Euler-Lagrange equations without necessarily being a minimizer, in this case $\bar{x}(.)$ is a critical point of $I[.]$. So, we can conclude that every minimizer is a critical point but a critical point need not be a minimizer.

5.4.3 Hamilton's ODE:

Suppose C^2 function $\bar{x}(.)$ is a critical point of the action functional and solves the Euler-Lagrange equations. Set

(1) $$\bar{p}(s)=D_qL\left(\dot{\bar{x}}(s),\bar{x}(s)\right) \qquad \left(0\le s\le t\right)$$

where $\bar{p}(.)$ is called the generalized momentum corresponding to the position $\bar{x}(.)$ and velocity $\dot{\bar{x}}(.)$.

Now we make important hypothesis:

(2) Hypothesis: Suppose for all $x,p\in R^n$ that the equation

$$p=D_qL(q,x)$$

can be uniquely solved for *q* as a smooth function of *p* and *x*, $q \quad =\bar{q}\ge\left(p,x\right)$

Definition: The Hamiltonian H associated with the Lagrangian L is

$$H\left(p,x\right)=p.\bar{q}\left(p,x\right)-L\left(\bar{q}\left(p,x\right),x\right) \qquad \left(p,x\in R^n\right)$$

where the function $\bar{q}(.,.)$ is defined implicitly by (2).

Example: The Hamiltonian corresponding to the Lagrangian $L(q,x)=\frac{1}{2}m|q|^2-\phi(x)$ is

$$H\left(p,x\right)=\frac{1}{2m}|p|^2+\phi(x)$$

The Hamiltonian is thus the total energy and the Lagrangian is the difference between the kinetic and potential energy.

5.4.4 Theorem: Derivative of Hamilton's ODE

The functions $\overline{x}(.)$ and $\overline{p}(.)$ satisfy Hamilton's equations

$$\text{(3)} \qquad \begin{cases} \dot{\overline{x}}(s) = D_p H\left(\overline{p}(s), \overline{x}(s)\right) \\ \dot{\overline{p}}(s) = -D_x H\left(\overline{p}(s), \overline{x}(s)\right) \end{cases} \qquad (0 \le s \le t)$$

Furthermore, the mapping $s \mapsto H\left(\overline{p}(s), \overline{x}(s)\right)$ is constant.

Proof: From (1) and (2), we have

$$\dot{\overline{x}}(s) = \overline{q}\left(\overline{p}(s), \overline{x}(s)\right)$$

Let us write $\overline{q}(.) = \left(q^1(.), ..., q^n(.)\right)$

We compute for $i = 1, ..., n$

$$\frac{\partial H}{\partial x_i}(p, x) = \sum_{k=1}^{n} p_k \frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x)\frac{\partial q^k}{\partial x_i}(p, x) - \frac{\partial L}{\partial x_i}(q, x)$$

$$= -\frac{\partial L}{\partial x_i}(q, x) \qquad \text{(using (2))}$$

and

$$\frac{\partial H}{\partial x_i}(p, x) = q^i(p, x) + \sum_{k=1}^{n} p_k \frac{\partial q^k}{\partial p_i}(p, x) - \frac{\partial L}{\partial q_k}(q, x)\frac{\partial q^k}{\partial p_i}(p, x)$$

$$= q^i(p, x) \qquad \text{(again using (2))}$$

Thus

$$\frac{\partial H}{\partial p_i}\left(\overline{p}(s), \overline{x}(s)\right) = q^i\left(\overline{p}(s), \overline{x}(s)\right) = \dot{x}^i(s)$$

and

$$\frac{\partial H}{\partial x_i}\left(\overline{p}(s), \overline{x}(s)\right) = -\frac{\partial L}{\partial x_i}\left(\overline{q}\left(\overline{p}(s), \overline{x}(s)\right), \overline{x}(s)\right) = -\frac{\partial L}{\partial x_i}\left(\dot{\overline{x}}(s), \overline{x}(s)\right)$$

$$= -\frac{d}{ds}\left(\frac{\partial L}{\partial q_i}\left(\dot{\overline{x}}(s), \overline{x}(s)\right)\right)$$

$$= -\dot{p}^i(s)$$

Hence

$$\frac{d}{ds} H\left(\overline{p}(s), \overline{x}(s)\right) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}^i + \frac{\partial H}{\partial x_i}\dot{x}^i$$

$$= \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\left(\frac{-\partial H}{\partial x_i}\right) + \frac{\partial H}{\partial x_i}\left(\frac{\partial H}{\partial p_i}\right) = 0$$

which shows that the mapping $s \to H(\overline{p}(s), \overline{x}(s))$ is constant.

5.5 Legendre transform:

Assume that the Lagrangian $L : R^n \to R$ satisfies following conditions

(i) the mapping $q \mapsto L(q)$ is convex …(1)

(ii) $\lim_{|q| \to \infty} \frac{L(q)}{|q|} = +\infty$ … (2)

whose convexity of the mapping in equation (2) implies L is continuous.

Note: In equation (2), we simplify the Lagrangian by dropping the x-dependence in the Hamiltonian so that afterwards *H=H(p).*

Definition: The Legendre transform of L is

(3) $$L^*(p) = \sup_{q \in R^n}\{p.q - L(q)\} \qquad (p \in R^n)$$

Remark: Hamiltonian H is the Legendre transform of L, and vice versa:

$$L = H^*, H = L^* \quad \dots (4)$$

We say H and L are dual convex functions.

Theorem: Convex duality of Hamiltonian and Lagrangian

Assume L satisfies (1),(2) and define H by (3),(4)

(i)Then

the mapping $p \mapsto H(p)$ is convex

And

$$\lim_{|p| \to \infty} \frac{H(p)}{|p|} = +\infty$$

(ii)Furthermore

$$L = H^* \quad \dots (5)$$

Proof: For each fixed q, the function $p \mapsto p.q - L(q)$ is linear, and the mapping

$$p \mapsto H(p) = L^*(p) = \sup_{q \in R^n}\{p.q - L(q)\}$$ is convex.

Indeed, if $0 \le \tau \le 1, p.\hat{p} \in R^n$,

$$H\left(\tau p+(1-\tau)\hat{p}\right)=\sup\left\{\left(\tau p+(1-\tau)\hat{p}\right).q-L(q)\right\}$$

$$\leq \tau \sup\left\{p.q-L(q)\right\}+(1-\tau)\sup_{q}\left\{\hat{p}.q-L(q)\right\}$$

$$=\tau H(p)+(1-\tau)H(\hat{p})$$

Fix any $\lambda > 0,\ p \neq 0$. Then

$$H(p)=\sup_{q\in R^n}\left\{p.q-L(q)\right\}$$

$$\geq \lambda|p|-L\left(\lambda\frac{p}{|p|}\right) \qquad \left(q=\lambda\frac{p}{|p|}\right)$$

$$\geq \lambda|p|-\max_{B(0,\lambda)} L$$

Therefore, $\liminf_{|p|\to\infty}\dfrac{H(p)}{|p|}\geq\lambda$ for all $\lambda>0$

From (4), we have

$$H(p)+L(q)\geq p.q \qquad \forall p,q\in R^n$$

and

$$L(q)\geq\sup_{p\in R^n}\left\{p.q-H(p)\right\}=H^*(q)$$

On the other hand

$$H^*(q)=\sup_{p\in R^n}\left\{p.q-\sup\left\{p.r-L(r)\right\}\right\}$$

$$=\sup_{p\in R^n}\inf_{r\in R^n}\left\{p.(q-r)+L(r)\right\} \qquad \dots\ (6)$$

since $q\mapsto L(q)$ is convex.

Let there exists $s\in R^n$ such that

$$L(r)\geq L(q)+s.(r-q) \qquad \left(r\in R^n\right)$$

Taking $p=s$ in (6)

$$H^*(q)\geq\inf_{r\in R^n}\left\{s.(q-r)+L(r)\right\}=L(q)$$

5.6 Hopf-Lax Formula

Consider the initial-value problem for the Hamilton-Jacobi equation

$$\begin{cases} u_t + H(Du) = 0 \; in \quad R^n \times (0,\infty) \\ \quad u = g \; on \quad R^n \times \{t = 0\} \end{cases} \qquad \dots (1)$$

We know that the calculus of variations problem with Lagrangian leads to Hamilton's ODE for the associated Hamilton H. Hence these ODE are also the characteristic equations of the Hamilton-Jacobi PDE, we infer there is probably a direct connection between this PDE and the calculus of variations.

Theorem: If $x \in R^n$ and $t > 0$, then the solution $u = u(x,t)$ of the minimization problem

$$u(x,t) = \inf\left\{ \int_0^t L(\dot{\bar{w}}(s))\,ds + g(y) \middle| \bar{w}(0) = y, \bar{w}(t) = x \right\} \qquad \dots (2)$$

is

$$u(x,t) = \min\left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\} \qquad \dots (3)$$

where, the infimum is taken over all C^1 functions. The expression on the right hand side of (3) called Hopf-Lax formula.

Proof: Fix any $y \in R^n$ and define

$$\bar{w}(s) = y + \frac{s}{t}(x-y) \qquad (0 \le s \le t)$$

Then $\bar{w}(0) = y$ and $\bar{w}(t) = y$

The expression (2) of u implies

$$u(x,t) \le \int_0^t L(\dot{\bar{w}}(s))\,ds + g(y) = tL\left(\frac{x-y}{t}\right) + g(y)$$

and therefore

$$u(x,t) \le \inf_{y \in R^n}\left\{ tL\left(\frac{x-y}{t}\right) + g(y) \right\}$$

If $\bar{w}(.)$ is any C^1 function satisfying $\bar{w}(t) = x$, then we have

$$L\left(\frac{1}{t}\int_0^t \dot{\bar{w}}(s)\,ds\right) \le \frac{1}{t}\int_0^t L(\dot{\bar{w}}(s))\,ds \qquad \text{(by Jensen's inequality)}$$

Thus if we write $y = w(0)$, we find

$$tL\left(\frac{x-y}{t}\right)+g(y)\leq\int_0^t L(\dot{w}(s))\,ds+g(y)$$

and consequently

$$\inf_{y\in R^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}\leq u(x,t)$$

Hence

$$u(x,t)=\inf_{y\in R^n}\left\{tL\left(\frac{x-y}{t}\right)+g(y)\right\}$$

Lemma 1: (A functional identity)

For each $x\in R^n$ and $0\leq s\leq t$, we have

$$u(x,t)=\min_{y\in R^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)\right\} \qquad \ldots\ (1)$$

In other words, to compute $u(.,t)$, we can calculate u at time s and then use $u(.,s)$ as the initial condition on the remaining time interval $[s,t]$.

Proof: Fix $y\in R^n, 0<s<t$ and choose $z\in R^n$ so that

$$u(y,s)=sL\left(\frac{y-z}{s}\right)+g(z) \qquad \ldots\ (2)$$

Now since L is convex and $\frac{x-z}{t}=\left(1-\frac{s}{t}\right)\left(\frac{x-y}{t-s}\right)+\frac{s}{t}\frac{y-z}{s}$, we have

$$L\left(\frac{x-z}{t}\right)\leq\left(1-\frac{s}{t}\right)L\left(\frac{x-y}{t-s}\right)+\frac{s}{t}L\left(\frac{y-z}{s}\right)$$

Thus

$$u(x,t)\leq tL\left(\frac{x-z}{t}\right)+g(z)\leq(t-s)L\left(\frac{x-y}{t-s}\right)+sL\left(\frac{y-z}{s}\right)+g(z)$$

$$=(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)$$

By (2). This inequality is true for each $y\in R^n$. Therefore, since $y\mapsto u(y,s)$ is continuous, we have

$$u(x,t)\leq\min_{y\in R^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right)+u(y,s)\right\} \qquad \ldots.\ (3)$$

Now choose w such that

$$u(x,t) = tL\left(\frac{x-w}{t}\right) + g(w) \qquad \ldots(4)$$

and set $y := \frac{s}{t}x + \left(1 - \frac{s}{t}\right)w$. Then $\frac{x-y}{t-s} = \frac{x-w}{t} = \frac{y-w}{s}$.

Consequently

$$(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)$$

$$\leq (t-s)L\left(\frac{x-w}{t}\right) + sL\left(\frac{y-w}{s}\right) + g(w)$$

$$= tL\left(\frac{x-w}{t}\right) + g(w) = u(x,t)$$

By (4). Hence

$$\min_{y \in R^n}\left\{(t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right\} \leq u(x,t) \qquad \ldots(5)$$

Lemma 2: (Lipschitz continuity)

The function u is Lipschitz continuous in $R^n \times [0,\infty)$, and $u = g$ on $R^n \times \{t = 0\}$.

Proof: Fix $t > 0, x, \hat{x} \in R^n$. Choose $y \in R^n$ such that

$$tL\left(\frac{x-y}{t}\right) + g(y) = u(x,t) \qquad \ldots.(6)$$

Then

$$u(\hat{x},t) - u(x,t) = \inf_z\left\{tL\left(\frac{\hat{x}-z}{t}\right) + g(z)\right\} - tL\left(\frac{x-y}{t}\right) - g(y)$$

$$\leq g(\hat{x} - x + y) - g(y) \leq Lip(g)|\hat{x} - x|$$

Hence

$$u(\hat{x},t) - u(x,t) \leq Lip(g)|\hat{x} - x|$$

and, interchanging the roles of $\hat{x}$ and x, we find

$$|u(\hat{x},t) - u(x,t)| \leq Lip(g)|x - \hat{x}| \qquad \ldots (7)$$

Now select $x \in R^n$, t>0. Choosing $y = x$ in (*), we discover

$$u(x,t) \le tL(0) + g(x) \qquad \ldots(8)$$

Furthermore,

$$u(x,t) = \min_{y \in R^n}\left\{ tL\left(\frac{x-y}{t}\right) + g(y)\right\}$$

$$\ge g(x) + \min_{y \in R^n}\left\{ -Lip(g)|x-y| + tL\left(\frac{x-y}{t}\right)\right\}$$

$$= g(x) - t\max_{z \in R^n}\{Lip(g)|z| - L(z)\} \qquad \left(z = \frac{x-y}{t}\right)$$

$$= g(x) - t\max_{w \in B(0,Lip(g))}\max_{z \in R^n}\{w.z - L(z)\}$$

$$= g(x) - t\max_{B(0,Lip(g))} H$$

This inequality and (8) imply

$$|u(x,t) - g(x)| \le Ct$$

For

$$\text{C} := \max\left(|L(0)|, \max_{B(0,Lip(g))}|H|\right) \qquad \ldots(9)$$

Finally select $x \in R^n, 0 < \hat{t} < t$. Then $Lip(u(.,t)) \le Lip(g)$ by (7) above. Consequently Lemma 1 and calculations like those employed in step 2 above imply

$$|u(x,t) - u(x,\hat{t})| \le C|t - \hat{t}|$$

For the constant C defined by (9).

Theorem: Solving the Hamilton-Jacobi equation

Suppose $x \in R^n, t > 0$, and u defined by the Hopf-Lax formula

$$u(x,t) = \min_{y \in R^n}\left\{ tL\left(\frac{x-y}{t}\right) + g(y)\right\}$$

is differentiable at a point $(x,t) \in R^n \times (0,\infty)$. Then

$$u_t(x,t) + H(Du(x,t)) = 0.$$

Proof: Fix $q \in R^n, h > 0$. Owing to Lemma 1,

$$u(x+hq,t+h) = \min_{y\in R^n}\left\{hL\left(\frac{x+hq-y}{h}\right)+u(y,t)\right\}$$
$$\leq hL(q)+u(x,t).$$

Hence

$$\frac{u(x+hq,t+h)-u(x,t)}{h} \leq L(q).$$

Let $h \to 0^+$, to compute

$$q.Du(x,t)+u_t(x,t) \leq L(q).$$

This inequality is valid for all $q \in R^n$, and so

$$u_t(x,t)+H(Du(x,t)) = u_t(x,t)+\max_{q\in R^n}\{q.Du(x,t)-L(q)\} \leq 0 \qquad \ldots(10)$$

The first equality holds since $H = L^*$.

Now choose z such that $u(x,t) = tL\left(\frac{x-z}{t}\right)+g(z)$. Fix h>0 and set $s = t-h, y = \frac{s}{t}x+\left(1-\frac{s}{t}\right)z$.

Then $\frac{x-z}{t} = \frac{y-z}{s}$, and thus

$$u(x,t)-u(y,s) \geq tL\left(\frac{x-z}{t}\right)+g(z)-\left[sL\left(\frac{y-z}{s}\right)+g(z)\right]$$
$$=(t-s)L\left(\frac{x-z}{t}\right)$$

That is,

$$\frac{u(x,t)-u\left(\left(1-\frac{h}{t}\right)x+\frac{h}{t}z,t-h\right)}{h} \geq L\left(\frac{x-z}{t}\right)$$

Let $h \to 0^+$ to compute

$$\frac{x-z}{t}.Du(x,t)+u_t(x,t) \geq L\left(\frac{x-z}{t}\right)$$

Consequently

$$u_t(x,t)+H(Du(x,t)) = u_t(x,t)+\max_{q\in R^n}\{q.Du(x,t)-L(q)\}$$
$$\geq u_t(x,t)+\frac{x-z}{t}.Du(x,t)-L\left(\frac{x-z}{t}\right)$$
$$\geq 0$$

This inequality and (10) complete the proof.

Lemma 3: (Semiconcavity)

Suppose there exists a constant C such that

$$g(x+z)-2g(x)+g(x-z)\le C|z|^2 \tag{11}$$

for all $x,z\in R^n$. Define u by the Hopf-Lax formula (*). Then

$$u(x+z,t)-2u(x,t)+u(x-z,t)\le C|z|^2$$

for all $x,z\in R^n, t>0$.

Remark: We say g is semiconcave provided (11) holds. It is easy to check (11) is valid if g is C^2 and $\sup_{R^n}|D^2 g|<\infty$. Note that g is semiconcave if and only if the mapping $x\mapsto g(x)+\frac{C}{2}|x|^2$ is concave for some constant C.

Proof: Choose $y\in R^n$ so that $u(x,t)=tL\left(\frac{x-y}{t}\right)+g(y)$. Then putting $y+z$ and $y-z$ in the Hopf-Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we find

$$\begin{aligned} &u(x+z,t)-2u(x,t)+u(x-z,t)\\ &\le\left[tL\left(\frac{x-y}{t}\right)+g(y+z)\right]-2\left[tL\left(\frac{x-y}{t}\right)+g(y)\right]\\ &\quad+\left[tL\left(\frac{x-y}{t}\right)+g(y-z)\right]\\ &=g(y+z)-2g(y)+g(y-z)\\ &\le C|z|^2, \qquad \text{by (11)} \end{aligned}$$

Definition: A C^2 convex function $H:R^n\to R$ is called uniformly convex(with constant $\theta>0$) if

$$\sum_{i,j=1}^{n} H_{p_i p_j}(p)\xi_i\xi_j\ge\theta|\xi|^2 \qquad \text{for all } p,\xi\in R^n \tag{12}$$

We now prove that even if g is not semi-concave, the uniform convexity of H forces u to become semi-concave for times t>0: it is a kind of mild regularizing effect for the Hopf-Lax solution of the initial- value problem.

Lemma 4: (Semi-concavity Again)

Suppose that H is uniformly convex (with constant θ) and u is defined by the Hopf-Lax formula. Then

$$u(x+z,t)-2u(x,t)+u(x-z,t)\le \frac{1}{\theta t}|z|^2$$

for all $x,z\in R^n, t>0$.

Proof: We note first using Taylor's formula that (12) implies

$$H\left(\frac{p_1+p_2}{2}\right)\le \frac{1}{2}H(p_1)+\frac{1}{2}H(p_2)-\frac{\theta}{8}|p_1-p_2|^2 \qquad (13)$$

Next we claim that for the Lagrangian L, we have estimate

$$\frac{1}{2}L(q_1)+\frac{1}{2}L(q_2)\le L\left(\frac{q_1+q_2}{2}\right)+\frac{1}{8\theta}|q_1-q_2|^2 \qquad (14)$$

For all $q_1,q_2\in R^n$. Verification is left as an exercise.

Now choose y so that $u(x,t)=tL\left(\frac{x-y}{t}\right)+g(y)$. Then using the same value of y in the Hopf-Lax formulas for $u(x+z,t)$ and $u(x-z,t)$, we calculate

$$\begin{aligned} &u(x+z,t)-2u(x,t)+u(x-z,t)\\ &\le\left[tL\left(\frac{x+z-y}{t}\right)+g(y)\right]-2\left[tL\left(\frac{x-y}{t}\right)+g(y)\right]\\ &\quad+\left[tL\left(\frac{x+z-y}{t}\right)+g(y)\right]\\ &=2t\left[\frac{1}{2}L\left(\frac{x+z-y}{t}\right)+\frac{1}{2}L\left(\frac{x-z-y}{t}\right)-L\left(\frac{x-y}{t}\right)\right]\\ &\le 2t\frac{1}{8\theta}\left|\frac{2z}{t}\right|^2\le\frac{1}{\theta t}|z|^2, \end{aligned}$$

The next-to-last inequality following from (14).

Theorem: Suppose $x\in R^n, t>0$, and u defined by the Hopf-Lax formula is differentiable at a point $(x,t)\in R^n\times(0,\infty)$. Then

$$u_t(x,t)+H(Du(x,t))=0$$

Proof: Fix $q \in R^n, h > 0$ and using Lemma (1), then we have

$$u(x+hq, t+h) = \min_{y \in R^n} \left\{ hL\left(\frac{x+hq-y}{h}\right) + u(y,t) \right\}$$

$$\leq hL(q) + u(x,t)$$

Hence

$$\frac{u(x+hq, t+h) - u(x,t)}{h} \leq L(q)$$

Let $h \to 0^+$, to compute

$$q.Du(x,t) + u_t(x,t) \leq L(q) \qquad \text{for all } q \in R^n$$

and therefore

$$u_t(x,t) + H(Du(x,t)) = u_t(x,t) + \max_{q \in R^n} \{q.Du(x,t) - L(q)\} \leq 0$$

The first equality holds since $H = L^*$

Now choose z such that

$$u(x,t) = tL\left(\frac{x-z}{t}\right) + g(z)$$

Fix h>0 and set

$$s = t-h, y = \frac{s}{t}x + \left(1 - \frac{s}{t}\right)z$$

Then $\dfrac{x-z}{t} = \dfrac{y-z}{s}$

and

$$u(x,t) - u(y,s) \geq tL\left(\frac{x-z}{t}\right) + g(z) - \left[sL\left(\frac{y-z}{s}\right) + g(z)\right]$$

$$= (t-s)L\left(\frac{x-z}{t}\right)$$

$$\Rightarrow \frac{u(x,t) - u\left(\left(1-\frac{h}{t}\right)x + \frac{h}{t}z, t-h\right)}{h} \geq L\left(\frac{x-z}{t}\right)$$

Let $h \to 0^+$ to compute

$$\frac{x-z}{t}.Du(x,t)+u_t(x,t)\geq L\left(\frac{x-z}{t}\right)$$

Consequently

$$u_t(x,t)+H(Du(x,t))=u_t(x,t)+\max_{q\in R^n}\{q.Du(x,t)-L(q)\}$$

$$\geq u_t(x,t)+\frac{x-z}{t}.Du(x,t)-L\left(\frac{x-z}{t}\right)$$

$$\geq 0$$

Hence $\qquad u_t(x,t)+H(Du(x,t))=0$

5 .7 Weak Solutions and Uniqueness

Definition: We say that a Lipschitz Continuous function $u:R^n\times[0,\infty)\to R$ is a weak solution of the initial-value problem

(15) $$\begin{cases} u_t+H(Du)=0\, in \quad R^n\times(0,\infty) \\ u=g\; on \quad R^n\times\{t=0\}\end{cases}$$

provided

(a) $u(x,0)=g(x) \qquad (x\in R^n)$

(b) $u_t(x,t)+H(Du(x,t))=0 \; for \quad a.e.(x,t)\in R^n\times(0,\infty)$

(c) $u(x+z,t)-zu(x,t)+u(x-z,t)\leq c\left(1+\frac{1}{t}\right)|z|^2$

for some constant $c\geq 0$ and all $x,z\in R^n, t>0$.

Theorem: Uniqueness of Weak Solution

Assume H is C^2 and satisfies $\begin{cases} H\; is \quad convex \quad and \\ \lim\limits_{|p|\to\infty}\frac{H(p)}{|p|}=+\infty\end{cases}$ and $g:R^n\to R$ is Lipschitz continuous. Then there exists at most one weak solution of the initial-value problem (15).

Proof: Suppose that u and $\tilde{u}$ are two weak solutions of (15) and write $w:=u-\tilde{u}$.

Observe now at any point (y,s) where both u and $\tilde{u}$ are differentiable and solve our PDE, we have

$$w_t(y,s)=u_t(y,s)-\tilde{u}_t(y,s)$$

$$= -H\left(Du(y,s)\right) + H\left(D\tilde{u}(y,s)\right)$$

$$= -\int_0^1 \frac{d}{dr} H\left(rDu(y,s) + (1-r)D\tilde{u}(y,s)\right) dr$$

$$= -\int_0^1 DH\left(rDu(y,s) + (1-r)D\tilde{u}(y,s)\right) dr.\left(Du(y,s) - D\tilde{u}(y,s)\right)$$

$$=: -b(y,s).Dw(y,s)$$

Consequently

$$w_t + b.Dw = 0 \quad \text{a.e.} \qquad \ldots (16)$$

Write $v := \phi(w) \geq 0$, where $\phi : R \to [0,\infty)$ is a smooth function to be selected later. We multiply(16) by $\phi'(w)$ to discover

$$v_t + b.Dv = 0 \quad \text{a.e.} \qquad \ldots (17)$$

Now choose $\varepsilon > 0$ and define $u^{\varepsilon} := \eta_{\varepsilon} * u, \tilde{u}^{\varepsilon} := \eta_{\varepsilon} * \tilde{u}$, where η_{ε} is the standard mollifier in the x and t variables. Then we have

$$\left|Du^{\varepsilon}\right| \leq Lip(u), \left|D\tilde{u}^{\varepsilon}\right| \leq Lip(\tilde{u}), \qquad \ldots (18)$$

and

$$Du^{\varepsilon} \to Du, D\tilde{u}^{\varepsilon} \to D\tilde{u} \quad \text{a.e., as } \varepsilon \to 0 \qquad \ldots (19)$$

Furthermore inequality(c) in the definition of weak solution implies

$$D^2 u^{\varepsilon}, D^2 \tilde{u}^{\varepsilon} \leq C\left(1 + \frac{1}{s}\right) I$$

For an appropriate constant C and all $\varepsilon > 0$, $y \in R^n$, $s > 2\varepsilon$. Verification is left as an exercise.

Write

$$b_{\varepsilon}(y,s) := \int_0^1 DH\left(rDu^{\varepsilon}(y,s) + (1-r)D\tilde{u}^{\varepsilon}(y,s)\right) dr \qquad \ldots (20)$$

Then (17) becomes

$$v_t + b_{\varepsilon}.Dv = (b_{\varepsilon} - b).Dv \quad \text{a.e.}$$

Hence

$$v_t + div(vb_{\varepsilon}) = (div b_{\varepsilon})v + (b_{\varepsilon} - b).Dv \quad \text{a.e.} \qquad \ldots (21)$$

Now

$$divb_\varepsilon = \int_0^1 \sum_{k,l=1}^n H_{p_k p_l}\left(rDu^\varepsilon + (1-r)D\tilde{u}^\varepsilon\right)\left(ru^\varepsilon_{x_l x_k} + (1-r)\tilde{u}^\varepsilon_{x_l x_k}\right)dr$$

$$\leq C\left(1+\frac{1}{s}\right) \qquad \ldots(22)$$

For some constant C, in view of (17) and (19). Here we note that H convex implies $D^2H \geq 0$.

Fix $x_0 \in R^n, t_0 > 0$, and set

$$R := \max\left\{\left|DH(p)\right| \middle| \left|p\right| \leq \max\left(Lip(\tilde{u})\right)\right\} \qquad \ldots(23)$$

Define also the cone

$$C := \left\{(x,t)\middle| 0 \leq t \leq t_0, \left|x-x_0\right| \leq R(t_0-t)\right\}$$

Next write

$$e(t) = \int_{B(x_0,R(t_0-t))} v(x,t)\,dx$$

and compute for a.e. t>0:

$$\dot{e}(t) = \int_{B(x_0,R(t_0-t))} v_t dx - R\int_{\partial B(x_0,R(t_0-t))} vdS$$

$$= \int_{B(x_0,R(t_0-t))} -div(vb_\varepsilon) + (divb_\varepsilon)v + (b_\varepsilon - b).Dvdx$$

$$-R\int_{\partial B(x_0,R(t_0-t))} vdS \qquad \text{by (21)}$$

$$= -\int_{\partial B(x_0,R(t_0-t))} v(b_\varepsilon.v + R)dS$$

$$+\int_{B(x_0,R(t_0-t))} (divb_\varepsilon)v + (b_\varepsilon - b).Dvdx$$

$$\leq \int_{B(x_0,R(t_0-t))} (divb_\varepsilon)v + (b_\varepsilon - b).Dvdx \qquad \text{by (17), (20)}$$

$$\leq C\left(1+\frac{1}{t}\right)e(t) + \int_{B(x_0,R(t_0-t))} (b_\varepsilon - b).Dvdx$$

by (22). The last term on the right hand side goes to zero as $\varepsilon \to 0$, for a.e. $t_0 > 0$, according to (17), (18) and the Dominated Convergence Theorem.

Thus

$$\dot{e}(t) \le C\left(1+\frac{1}{t}\right)e(t) \qquad \text{for a.e. } 0 < t < t_0 \tag{24}$$

Fix $0 < \varepsilon < r < t$ and choose the function $\phi(z)$ to equal zero if

$$|z| \le \varepsilon\left[Lip(u) + Lip(\tilde{u})\right]$$

and to be positive otherwise. Since $u = \tilde{u}$ on $R^n \times \{t = 0\}$,

$$v = \phi(w) = \phi(u - \tilde{u}) = 0 \quad \text{at } \{t = \varepsilon\}$$

Thus $e(\varepsilon) = 0$. Consequently Gronwall's inequality and (24) imply

$$e(r) \le e(\varepsilon)e^{\int_\varepsilon^r C\left(1+\frac{1}{s}\right)dS} = 0$$

Hence

$$|u - \tilde{u}| \le \varepsilon\left[Lip(u) + Lip(\tilde{u})\right] \quad \text{on } B\big(x_0, R(t_0 - r)\big)$$

This inequality is valid for all $\varepsilon > 0$, and so $u \equiv \tilde{u}$ in $B\big(x_0, R(t_0 - r)\big)$. Therefore, in particular, $u(x_0, t_0) = \tilde{u}(x_0, t_0)$.